

# Aspects of high density effective theory in QCD

Deog Ki Hong\*

*Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138*

*and*

*Physics Department, Boston University, Boston, MA 02215, USA*

## Abstract

We study an effective theory of QCD at high density in detail, including the finite temperature effects and the leading order correction in  $1/\mu$  expansion. We investigate the Cooper pair gap equation and find that the color-flavor locking phase is energetically preferred at high density. We also find the color-superconducting phase transition occurs in dense quark matter when the chemical potential is larger than  $250 \pm 100$  MeV and the temperature is lower than 0.57 times the Cooper pair gap in the leading order in the hard-dense-loop approximation. The quark-neutrino four-Fermi coupling and the quark-axion coupling receive significant corrections in dense quark matter.

PACS: 12.20.Ds, 12.38.Mh, 11.10.Gh

*Keywords:* Finite density/temperature QCD; Effective Theory; Cooper pair

Typeset using REVTeX

---

\*On leave from Department of Physics, Pusan National University, Pusan 609-735, Korea.  
E-mail address: dkhong@hyowon.cc.pusan.ac.kr

## I. INTRODUCTION

Effective field theory is indispensable to describe physical processes at energy lower than the characteristic scale of theory, especially when the microscopic physics is too complicated to be used as in the case of chiral perturbation theory [1] or just unknown as in the standard model. Since effective field theory has been very useful and successful in explaining the low energy data, it has now become an everyday language in physics [2,3].

Color superconductivity in cold and dense quark matter [4] has been studied quite intensively in recent years by using effective field theory methods [5–9]. Most important ingredient in those studies was the existence of effective four-quark interactions that mimic the four-fermion interaction generated by phonon exchange in BCS superconductivity, which are assumed to be generated in the effective Lagrangian either by exchange of massive gluons [8] or by strongly coupled instantons [5,6]. For low density quark matter, where strong interaction is no longer weak, such four-Fermi interactions may describe QCD at low density as model interactions, thus leading to color superconducting gap of QCD scale ( $\sim \Lambda_{\text{QCD}}$ ). But, it was recently argued quite convincingly by Son [10] that at least for high density quark matter, where perturbative QCD is applicable, magnetic gluons are not Debye-screened but only dynamically screened or Landau damped. Such long-range magnetic gluon interaction is more important and leads to a bigger Cooper-pair gap,  $\Delta \sim \mu g_s^{-5} \exp(-3\pi^2/\sqrt{2}g_s)$ , than the usual BCS gap, which is subsequently confirmed by the Schwinger-Dyson analysis for the Cooper-pair gap [11–14].

Recently, an effective field theory of QCD at high density [11] is derived systematically in the power expansion of  $1/\mu$  by integrating out the anti-quarks which are decoupled at low energy in dense quark matter. In the effective theory, four-quark operators are generated at the one-loop matching and the electric gluons are screened due to quarks in the Fermi sea while the magnetic gluons are not at least in perturbation [10,15–18]. At scales below the electric screening mass, the relevant interactions for quarks are the coupling with magnetic gluons and the four-quark interaction with opposite momenta. Both interactions for quarks in the color anti-triplet channel with opposite momenta are shown to lead to color superconductivity [10,11].

According to the quark-hadron phase continuity in QCD, conjectured recently by Schäfer and Wilczek [19], the confining phase of nuclear matter at low density is complementary to the Higgs phase quark matter at high density. We support this conjecture by showing that in the effective theory, which is equivalent to QCD in the asymptotic density, the color-flavor locking diquark condensate is energetically more preferred. Furthermore, in the effective theory, the higher order corrections are systematically calculable and one can easily estimate quantities like the critical density, which are determined by the sub-leading operators in  $1/\mu$  expansion.

In this article, we study the effective lagrangian of QCD at high density derived in [11] in detail, including the higher order corrections in  $1/\mu$  expansion and the finite temperature effects. We then analyze the Cooper-pair gap equation more rigorously and calculate the gap, the critical temperature, and the critical density. As applications, we examine the effect of marginal four-quark operators on the neutrino-quark four-Fermi coupling and the quark-axion coupling in super-dense quark matter.

## II. HIGH DENSITY EFFECTIVE THEORY

The idea of effective field theory is quite simple and its construction can be performed in a systematic way. First, one identifies or guesses the right degrees of freedom to describe the low energy processes. Then, one integrates out the irrelevant degrees of freedom, which results in nonlocal interactions. By expanding the nonlocal interactions in powers of momentum at low energy, one derives a (Wilsonian) effective Lagrangian, which is usually done in practice by matching all the one light-particle irreducible amplitudes in the full theory with those amplitudes in the effective theory in loop expansion. Since the higher dimensional operators have smaller effects at low energy, the effective theories provide a very useful approach to describe low energy physics.

A system of degenerate quarks with a fixed baryon number is described by the QCD Lagrangian density with a chemical potential  $\mu$ ,

$$\mathcal{L}_{\text{QCD}} = \bar{\psi} i \not{D} \psi - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \mu \bar{\psi} \gamma_0 \psi, \quad (2.1)$$

where the covariant derivative  $D_\mu = \partial_\mu + i g_s A_\mu^a T^a$  and we neglect the mass of quarks for simplicity. The chemical potential is introduced as a Lagrangian multiplier to constrain the system to have a fixed baryon number,  $N_B$ . At zero temperature, all the states up to the Fermi surface are filled;

$$\frac{N_B}{V} = 2N_f \int \frac{d^3p}{(2\pi)^3} \theta(p_F - |\vec{p}|), \quad (2.2)$$

where  $V$  is the volume,  $N_f$  the number of quark flavors, and  $\vec{p}_F$  is the Fermi momentum. The chemical potential  $\mu$ , which defines the energy at the Fermi surface, provides another scale for QCD. Here we consider a super-dense quark matter where quarks are very dense so that the average inter-quark distance is much smaller than the characteristic length of QCD,  $1/\mu \ll 1/\Lambda_{\text{QCD}}$ . For high chemical potential, the pair creation of a particle and an anti-particle is suppressed at low energy because of the energy gap  $\mu$ , provided by the Fermi sea. Therefore, the relevant degrees of freedom at low energy will be just particles and holes, excited near the Fermi surface, together with gluons.

We will derive a low energy effective Lagrangian for cold and dense quark matter by integrating out the high frequency or irrelevant modes, *à la* Wilson, as we run QCD from high energy  $E > \mu$  to low energy  $E \sim \Lambda_{\text{QCD}}$ . Since QCD is asymptotically free, quarks will interact weakly at energy  $E > \mu$  ( $\gg \Lambda_{\text{QCD}}$ ) and their spectrum will be approximately given by the free Hamiltonian:

$$(\vec{\alpha} \cdot \vec{p} - \mu) \psi_\pm = E_\pm \psi_\pm, \quad (2.3)$$

where  $\vec{\alpha} = \gamma_0 \vec{\gamma}$  and  $\psi_\pm$  denote the energy eigenfunctions with eigenvalues  $E_\pm = -\mu \pm |\vec{p}|$ , satisfying  $\vec{\alpha} \cdot \vec{p} \psi_\pm = \pm |\vec{p}| \psi_\pm$ , respectively. At low energy  $E < \mu$ , the states  $\psi_+$  near the Fermi surface,  $|\vec{p}| \sim \mu$ , are easily excited, while  $\psi_-$ , which correspond to the states in the Dirac sea, are completely decoupled due to the presence of the energy gap  $\mu$  provided by the Fermi sea. Therefore the right degrees of freedom below  $\mu$  consist of gluons and  $\psi_+$  only.

If  $\mu = 0$ , integrating out the high frequency modes in QCD just renormalizes the coupling constants without generating any new interactions as it should be for a renormalizable

theory like QCD. However, when  $\mu \neq 0$ , the energy spectra of particles and anti-particles are asymmetric. Anti-particles ( $\bar{\psi}_-$ ), which correspond to the absence of states ( $\psi_-$ ) in the Dirac sea, have energy larger than  $\mu$  and will be decoupled at energy scale lower than  $\mu$ . Therefore, the minimal quark-gluon coupling that connects  $\psi_-$  with  $\bar{\psi}_+$  (or vice versa), has to be removed at low energy, generating a new interaction among  $\psi_+$ . We will see later that integrating out the modes whose frequency is higher than  $\mu$  results in an effective Lagrangian which is drastically different from QCD.

We now integrate out the antiparticles ( $\bar{\psi}_-$ ) or the states ( $\psi_-$ ) in the Dirac sea to derive the low energy effective Lagrangian. We first decompose the quark momentum into the Fermi momentum and a residual momentum as

$$p_\mu = \mu v_\mu + l_\mu, \quad (2.4)$$

where the residual momentum  $|l_\mu| < \mu$  and  $v^\mu = (0, \vec{v}_F)$  with Fermi velocity  $\vec{v}_F$ . The magnitude of Fermi velocity,  $|\vec{v}_F| \equiv p_F / \sqrt{p_F^2 + m^2}$ , will be taken to be unity, for we assume the quark mass  $m = 0$ .

Since we will integrate out the high frequency modes such that we will be left with excitations only near the Fermi surface at low energy, it is convenient to decompose the quark field by the Fermi velocity <sup>1</sup>

$$\psi(x) = \sum_{\vec{v}_F} e^{i\mu\vec{v}_F \cdot \vec{x}} \psi(\vec{v}_F, x), \quad (2.5)$$

where

$$\psi(\vec{v}_F, x) = \int_{|l_\mu| < \mu} \frac{d^4 l}{(2\pi)^4} \psi(\vec{v}_F, l) e^{-il \cdot x} \quad (2.6)$$

carries the residual momentum  $l_\mu$ . We introduce projection operators  $P_\pm = (1 \pm \vec{\alpha} \cdot \vec{v}_F)/2$  and define

$$\psi_\pm(\vec{v}_F, x) \equiv P_\pm \psi(\vec{v}_F, x). \quad (2.7)$$

In the limit  $l/\mu \rightarrow 0$ ,  $\psi_-(\vec{v}_F, x)$  corresponds to the states in the Dirac sea and  $\psi_+(\vec{v}_F, x)$  to the states above the Dirac sea.

In terms of the new fields, the QCD Lagrangian for quarks at high density becomes

$$\begin{aligned} \mathcal{L}_{\text{quark}} &= \bar{\psi}(x) \left( i \not{D} + \mu \gamma^0 \right) \psi(x) \\ &= \sum_{\vec{v}_F} \left[ \bar{\psi}_+(\vec{v}_F, x) i \gamma^0 V \cdot D \psi_+(\vec{v}_F, x) + \bar{\psi}_-(\vec{v}_F, x) \gamma^0 \left( 2\mu + i \vec{V} \cdot D \right) \psi_-(\vec{v}_F, x) \right. \\ &\quad \left. + \bar{\psi}_-(\vec{v}_F, x) i \gamma_\perp^\mu D_\mu \psi_+(\vec{v}_F, x) + \bar{\psi}_+(\vec{v}_F, x) i \gamma_\perp^\mu D_\mu \psi_-(\vec{v}_F, x) \right], \end{aligned} \quad (2.8)$$

---

<sup>1</sup>For a given quark momentum, the corresponding Fermi velocity is determined up to reparametrization;  $\vec{v}_F \rightarrow \vec{v}_F + \delta \vec{l}_\perp / \mu$  and  $\vec{l} \rightarrow \vec{l} - \delta \vec{l}$ , where  $\delta \vec{l}_\perp$  is a residual momentum perpendicular to the Fermi velocity. As in the heavy quark effective theory [20], the renormalization of higher-order operators are restricted due to this reparametrization invariance.

where  $V^\mu = (1, \vec{v}_F)$ ,  $\bar{V}^\mu = (1, -\vec{v}_F)$ ,  $\gamma_\perp^\mu = \gamma^\mu - \gamma_\parallel^\mu$  with  $\gamma_\parallel^\mu = (\gamma^0, \vec{v}_F \vec{v}_F \cdot \vec{\gamma})$ , and we have used

$$\begin{aligned}\bar{\psi}_+(\vec{v}_F, x) \gamma^\mu \psi_+(\vec{v}_F, x) &= \bar{\psi}_+ P_- \gamma^\mu P_+ \psi_+ = V^\mu \bar{\psi}_+ \gamma^0 \psi_+ \\ \bar{\psi}_-(\vec{v}_F, x) \gamma^\mu \psi_-(\vec{v}_F, x) &= \bar{\psi}_- P_+ \gamma^\mu P_- \psi_- = \bar{V}^\mu \bar{\psi}_- \gamma^0 \psi_- \\ \bar{\psi}_+(\vec{v}_F, x) \gamma^\mu \psi_-(\vec{v}_F, x) &= \bar{\psi}_+ P_- \gamma^\mu P_- \psi_- = \bar{\psi}_+ \gamma_\perp^\mu \psi_- \\ \bar{\psi}_-(\vec{v}_F, x) \gamma^\mu \psi_+(\vec{v}_F, x) &= \bar{\psi}_- P_+ \gamma^\mu P_+ \psi_+ = \bar{\psi}_- \gamma_\perp^\mu \psi_+.\end{aligned}\tag{2.9}$$

In Eq. (2.8) the incoming quark and the outgoing quark have the same Fermi velocity  $\vec{v}_F$ , since, at low energy, the momentum carried away by gluons can be compensated by the residual momentum of the quarks to conserve the momentum, without changing the Fermi velocity.

As we can see from Eq. (2.8), the propagation of quarks at high density is 1+1 dimensional if  $l_\perp^i/\mu$  is negligible. This dimensional reduction at high density can be also seen if we decompose the quark propagator as following:

$$\begin{aligned}iS_F(p) &= \frac{i}{(1 + i\epsilon)p^0 \gamma^0 - \vec{p} \cdot \vec{\gamma} + \mu \gamma^0} \\ &= \frac{\not{V}}{2} \frac{i}{l \cdot V + i\epsilon l_0} + \frac{\not{\bar{V}}}{2} \frac{i}{2\mu + l \cdot \bar{V} + i\epsilon l_0} + O(l_\perp^i/\mu),\end{aligned}\tag{2.10}$$

where quark momentum  $p^\mu = \mu v^\mu + l^\mu$  and the  $i\epsilon$  prescription is chosen such that not only the anti-particles but also the holes correspond to the negative energy states moving backward in time. Note also that, if  $l_\perp^i/\mu \rightarrow 0$ ,  $1/2 \not{V} \gamma^0 = 1/2(1 + \vec{\alpha} \cdot \vec{v}_F)$  is the projection operator that projects out particles and holes ( $\psi_+$ ), while  $1/2 \not{\bar{V}} \gamma^0 = 1/2(1 - \vec{\alpha} \cdot \vec{v}_F)$  projects out the states in the Dirac sea ( $\psi_-$ ). Since the excitation of quarks along the perpendicular direction to the quark velocity does not cost any energy in the leading order, the perpendicular momentum is nothing but an index that labels the degeneracy in quark just as the degeneracy in the Landau level under external constant magnetic field is labeled by the electron momentum perpendicular to the external magnetic field.

At tree-level, integrating out the irrelevant modes  $\psi_-(\vec{v}_F, x)$  is tantamount to eliminating  $\psi_-(\vec{v}_F, x)$  by the equations of motion, given as

$$\psi_-(\vec{v}_F, x) = -\frac{i\gamma^0}{2\mu + i\bar{D}_\parallel} \not{D}_\perp \psi_+(\vec{v}_F, x) = -\frac{i\gamma^0}{2\mu} \sum_{n=0}^{\infty} \left( -\frac{i\bar{D}_\parallel}{2\mu} \right)^n \not{D}_\perp \psi_+(\vec{v}_F, x),\tag{2.11}$$

where  $\bar{D}_\parallel = \bar{V}^\mu D_\mu$  and  $\not{D}_\perp = \gamma_\perp^\mu D_\mu$ . Plugging Eq. (2.11) into the Lagrangian for quarks Eq. (2.8), we obtain the tree-level effective Lagrangian of QCD at high density,

$$\mathcal{L}_{eff}^0 = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \sum_{\vec{v}_F} \left[ \psi_+^\dagger(\vec{v}_F, x) iV \cdot D \psi_+(\vec{v}_F, x) - \psi_+^\dagger \frac{(\not{D}_\perp)^2}{2\mu} \sum_{n=0}^{\infty} \left( -\frac{i\bar{D}_\parallel}{2\mu} \right)^n \psi_+(\vec{v}_F, x) \right].\tag{2.12}$$

### III. ONE-LOOP MATCHING AND A FOUR-FERMI OPERATOR

As we have seen in the previous section, integrating out the irrelevant modes generates a new coupling between quarks and gluons. This can be also seen diagrammatically as shown

in Fig. 1: In the leading order, the propagator of fast modes  $\psi_-$  is replaced by a constant matrix

$$\frac{i\gamma^0}{2\mu + i\bar{D}_\parallel} = \frac{i\gamma^0}{2\mu} + O(l/\mu) \quad (3.1)$$

and the exchange of  $\psi_-$  generates a new interaction given as, using  $P_- \gamma^\mu P_- = P_- \gamma_\perp^\mu$ ,

$$\mathcal{L}_{\text{eff}}^0 \ni -\frac{g^2}{2\mu} \sum_{\vec{v}_F} \bar{\psi}_+ A_\perp \gamma_0 A_\perp \psi_+ + \dots, \quad (3.2)$$

where the ellipsis denotes terms containing more powers of gluons and derivatives. By writing the interaction in a gauge-invariant fashion, we recover the interaction terms in Eq. (2.12).

The effect of integrating out fast modes not only generates new interactions at low energy but also gives rise to quantum corrections to the tree-level couplings which can be calculated explicitly by matching the loop amplitudes. Usually the new interactions arise at tree-level upon integrating out the fast modes, but sometimes they may arise at quantum level, especially when they are marginal, as in the low energy effective theory of QED under external (strong) magnetic field [21].

As fermions under external fields [21–23], in the presence of a Fermi sea [3] the scaling dimension of fields at low energy changes due to the change in the particle spectrum; the scaling dimension of quarks becomes  $-1/2$  instead of  $-3/2$ , the canonical scaling dimension of fermions in 3+1 dimensions. The scaling dimension of fields is determined by the kinetic term in the action, which is for quarks in dense quark matter given in the high density limit as

$$S_0 = \sum_{\vec{v}_F} \int \frac{d^4l}{(2\pi)^4} \left[ \psi_+^\dagger(\vec{v}_F, l) (l_0 - v_F l_\parallel) \psi_+(\vec{v}_F, l) + \psi_-^\dagger(\vec{v}_F, l) (2\mu + l \cdot \vec{V}) \psi_-(\vec{v}_F, l) \right], \quad (3.3)$$

where  $l_\parallel \equiv \vec{v}_F \cdot \vec{l}$  is the residual momentum parallel to the Fermi velocity. Under a scale transformation  $l_0 \rightarrow s l_0$  and  $l_\parallel \rightarrow s l_\parallel$  with  $s < 1$ , the quark fields transform as  $\psi_+(\vec{v}_F, l) \rightarrow s^{-3/2} \psi_+(\vec{v}_F, l)$  and  $\psi_-(\vec{v}_F, l) \rightarrow s^{-1} \psi_-(\vec{v}_F, l)$ . As we scale the momentum toward the Fermi surface or we decrease  $s \rightarrow 0$ ,  $\psi_+$  scales like free massless fermions in (1+1)-dimensions, while  $\psi_-$  scales like heavy fermions in (1+1)-dimensions and becomes irrelevant at low energy. Since four-Fermi interactions are marginal for (1+1)-dimensional (light) fermions, we expect that a marginal four-quark operator will arise in the high density effective theory of QCD by quantum effects if it is absent at the tree-level. To see this, we consider one-loop matching of a four-quark amplitude for scattering of quarks with opposite Fermi velocities, shown in Fig. 2, because only when the incoming quarks have opposite Fermi momenta the four-quark operators are marginal as we scale toward the Fermi surface [3]. Since the effective theory amplitude is ultra-violet (UV) divergent while the amplitude in full QCD is UV finite by the power counting, we need a UV counter term for the one-loop matching in the effective theory, which is nothing but a four-quark operator. The one-loop four-quark amplitude of our interest in the effective theory is, reverting the notation  $\psi$  for  $\psi_+$  henceforth,

$$A_4^{\text{eff}} = \frac{1}{2} \left( -\frac{ig_s^2}{2\mu} \right)^2 \sum_{\vec{v}_F, \vec{v}'_F} \langle \vec{v}_3, l_3; \vec{v}_4, l_4 | \int_{x,y} \bar{\psi} A_\perp \gamma^0 A_\perp \psi(\vec{v}_F, x) \cdot \bar{\psi} A_\perp \gamma^0 A_\perp \psi(\vec{v}'_F, y) | \vec{v}_1, l_1; \vec{v}_2, l_2 \rangle$$

$$\begin{aligned}
&= -\frac{g_s^4}{4\mu^2} (2\pi)^4 \delta^4 \left( \sum_i' p_i \right) \bar{\psi}(p_3) \gamma_\perp^\mu \gamma_\perp^0 \gamma_\perp^\nu T^a T^b \psi(p_1) \cdot \bar{\psi}(p_4) \gamma_\perp^\rho \gamma_\perp^0 \gamma_\perp^\sigma T^c T^d \psi(p_2) \\
&\times \left( \delta^{ac} \delta^{bd} g_{\mu\rho} g_{\nu\sigma} + \delta^{ad} \delta^{bc} g_{\mu\sigma} g_{\nu\rho} \right) \cdot \frac{-i}{16\pi^2} \left[ \frac{1}{\epsilon} - \gamma - 2 - \ln \left( \frac{-l^2 - i\epsilon}{4\pi\Lambda^2} \right) \right],
\end{aligned} \tag{3.4}$$

where  $\vec{v}_i$ 's are the Fermi velocities of incoming and outgoing particles with  $\vec{v}_1 = -\vec{v}_2$ ,  $p_i = \mu\vec{v}_i + l_i$ ,  $\gamma$  the Euler number,  $\sum_i' p_i \equiv p_3 + p_4 - p_1 - p_2$ ,  $l = l_3 - l_1$ , and  $\Lambda$  is the renormalization point.

In QCD, the quark-quark scattering amplitude is

$$\begin{aligned}
A_4^{\text{QCD}} &= \frac{g_s^4}{4!} \langle p_3, p_4 | \left[ \sum_{\vec{v}_F} \int_x \bar{\psi}(\vec{v}_F, x) A_\perp(x) \psi(\vec{v}_F, x) \right]^4 | p_1, p_2 \rangle \\
&= g_s^4 (2\pi)^4 \delta^4 \left( \sum_i' p_i \right) \psi^\dagger(p_3) \gamma_\perp^\mu \gamma_\perp^\nu T^a T^b \psi(p_1) \psi^\dagger(p_4) \left[ I_1 \gamma_\mu \gamma_\nu T^a T^b + I_2 \gamma_\nu \gamma_\mu T^b T^a \right] \psi(p_2)
\end{aligned} \tag{3.5}$$

where

$$I_1 = \int_q \frac{1}{2\mu + \bar{V} \cdot (l_3 + q)} \frac{1}{2\mu + V \cdot (l_4 - q)} \frac{1}{(q + l)^2 q^2} \tag{3.6}$$

$$I_2 = \int_q \frac{1}{2\mu + \bar{V} \cdot (l_1 - q)} \frac{1}{2\mu + V \cdot (l_4 - q)} \frac{1}{(q + l)^2 q^2} \tag{3.7}$$

In the limit that the residual momenta  $l_i \rightarrow 0$ , we find  $I_1 = I_2$ , if we rotate the  $q_0$  axis into  $\vec{v}_F \cdot \vec{q}$  axis, and thus the color and Lorentz structures of the amplitudes in both theories are same. To perform the integration, we take, for convenience, the external residual momenta are perpendicular to the Fermi velocity,  $V \cdot l_i = 0$ . Then, we get for  $l_i/\mu \rightarrow 0$

$$\begin{aligned}
I_1 &= \int_0^1 dx \int_0^{1-x} dy \int_q \frac{-2}{[(1-x-y)q^2 + x(q+l)^2 + yq_0^2 - y(\vec{v}_F \cdot \vec{q} + 2\mu)^2]^3} \\
&= \frac{i}{64\pi^2 \mu^2} \left[ 2 - \ln \left( \frac{-l^2}{4\mu^2} \right) + O(l^2/\mu^2) \right].
\end{aligned} \tag{3.8}$$

Note that the scattering amplitude  $A_4^{\text{QCD}}$  is IR divergent as  $l_i \rightarrow 0$ , which is the same infrared divergence of the amplitude in the effective theory, Eq. (3.4), as it should be because the low energy effective theory is equivalent to the microscopic theory in the infrared limit by construction. Since both amplitudes have to be same at the matching scale  $\Lambda = \mu$ , we need a four-quark operator in the effective theory, given by the difference of the amplitudes at the matching scale  $\mu$ :

$$\begin{aligned}
A_4^{\text{QCD}} - A_4^{\text{eff}} &= \frac{ig_s^4}{64\pi^2 \mu^2} \left( \frac{1}{\epsilon} - \gamma - 4 \right) (2\pi)^4 \delta^4 \left( \sum_i' p_i \right) \psi^\dagger(p_3) \gamma_\perp^\mu \gamma_\perp^\nu T^a T^b \psi(p_1) \\
&\times \psi^\dagger(p_4) \left( \gamma_\mu \gamma_\nu T^a T^b + \gamma_\nu \gamma_\mu T^b T^a \right) \psi(p_2) + O(l_i/\mu) \\
&= \langle p_3, p_4 | i \int d^4x \mathcal{L}_{4f}^1 | p_1, p_2 \rangle + O(l_i/\mu),
\end{aligned} \tag{3.9}$$

where, at the matching scale  $\mu$ , the new effective four-quark operator <sup>2</sup> generated at one-loop is given as

$$\mathcal{L}_{4f}^1 = -\frac{g_1^{\text{bare}}}{2\mu^2} \sum_{\vec{v}_F} \left[ \psi^\dagger(\vec{v}_F, x) \gamma_\perp^\mu \gamma_\perp^\nu T^a T^b \psi(\vec{v}_F, x) \right. \\ \left. \times \psi^\dagger(-\vec{v}_F, x) \left( \gamma_\mu \gamma_\nu T^a T^b + \gamma_\nu \gamma_\mu T^b T^a \right) \psi(-\vec{v}_F, x) \right]. \quad (3.10)$$

In the modified minimal subtraction, the renormalized four-quark coupling is  $g_1^{\text{ren}} = 2\alpha_s^2$  at the matching scale  $\mu$ . It is convenient to rewrite the marginal four-quark operator by Fierz-transforming the product of gamma matrices and  $SU(3)_C$  generators in the amplitude Eq. (3.10). Using  $(T^a)_{tu} (T^a)_{vs} = 1/2\delta_{ts}\delta_{uv} - 1/6\delta_{tu}\delta_{vs}$  and the Fierz transformation, we get

$$\begin{aligned} & [P_+ \gamma_\perp^\mu \gamma_\perp^\nu P_+]_{ji} (T^a T^b)_{ut} \cdot [P_- (\gamma_{\mu\perp} \gamma_{\nu\perp} T^a T^b + \gamma_{\nu\perp} \gamma_{\mu\perp} T^b T^a) P_-]_{lm;vs} \\ &= [P_+ \gamma_\perp^\mu \gamma_\perp^\nu P_+]_{ji} (T^a T^b)_{ut} \cdot \left[ g_{\mu\nu} P_- \left( \frac{1}{3} \delta^{ab} + d^{abc} T^c \right) + (P_- \sigma_{\mu\nu} P_-) f^{abc} T^c \right]_{lm,vs} \\ &= (P_+)_{ji} (P_-)_{lm} \left( \frac{11}{18} \delta_{ut} \delta_{vs} + \frac{5}{6} \delta_{us} \delta_{tv} \right) + (P_+ \gamma_5)_{ji} (P_- \gamma_5)_{lm} \left( -\frac{1}{2} \delta_{ut} \delta_{vs} + \frac{3}{2} \delta_{us} \delta_{tv} \right) \\ &= - (P_+)_{ji} (P_-)_{lm} \left( \frac{\sqrt{2}}{9} \delta_{uv;ts}^A - \frac{13}{9\sqrt{2}} \delta_{uv;ts}^S \right) - (P_+ \gamma_5)_{ji} (P_- \gamma_5)_{lm} \left( \sqrt{2} \delta_{uv;ts}^A - \frac{1}{\sqrt{2}} \delta_{uv;ts}^S \right), \end{aligned} \quad (3.11)$$

where  $i, j, l, m$  denote the Dirac indices and  $t, s, u, v$  the color indices. In the last line, the color indices in the operator are further arranged to decompose the amplitude into the irreducible representations of  $SU(3)_C$  by introducing invariant tensors in color space,  $\delta_{uv;ts}^S \equiv (\delta_{ut} \delta_{vs} + \delta_{us} \delta_{vt}) / \sqrt{2}$  and  $\delta_{uv;ts}^A \equiv (\delta_{ut} \delta_{vs} - \delta_{us} \delta_{vt}) / \sqrt{2}$ .

The four-quark operator in the effective theory becomes then

$$\mathcal{L}_{4f}^1 = \frac{1}{2\mu^2} \sum_{\vec{v}_F} \left[ g_{us;tv} \psi_t^\dagger(\vec{v}_F, x) \psi_s(\vec{v}_F, x) \psi_v^\dagger(-\vec{v}_F, x) \psi_u(-\vec{v}_F, x) \right. \\ \left. + h_{us;tv} \psi_t^\dagger(\vec{v}_F, x) \gamma_5 \psi_s(\vec{v}_F, x) \psi_v^\dagger(-\vec{v}_F, x) \gamma_5 \psi_u(-\vec{v}_F, x) \right] \quad (3.12)$$

with  $g_{us;tv} = g_3 \delta_{us;tv}^A - g_6 \delta_{us;tv}^S$ ,  $h_{us;tv} = h_3 \delta_{us;tv}^A - h_6 \delta_{us;tv}^S$ . The value of couplings at the matching scale  $\mu$  are given as  $g_3 = 4\sqrt{2}\alpha_s^2/9 = 2g_6/(13)$  and  $h_3 = 4\sqrt{2}\alpha_s^2 = 2h_6$ .

#### IV. SCREENING MASS

As discussed in [16–18,24], the quark loop correction to the vacuum polarization tensor gives rise to color screening in quark-gluon plasma, while the gluon loop renormalizes the color charge. We first calculate the vacuum polarization tensor in QCD and then compare

---

<sup>2</sup> If we had matched a quark-quark scattering amplitude with momenta not opposite to each other, we would get similarly a four-quark operator in the effective Lagrangian. But, since it is irrelevant as we scale toward the Fermi sea, it will not affect the low energy dynamics significantly.

it with the one in the effective theory when we perform the one-loop matching for the gluon two-point amplitude.

Since the Feynman propagator for quarks of mass  $m$  in matter at zero temperature is given by

$$iS_F(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot x} \frac{i}{(1+i\epsilon)p_0\gamma^0 - \vec{p} \cdot \vec{\gamma} + \mu\gamma^0 - m}, \quad (4.1)$$

we find, performing the integration over  $p_0$ ,

$$iS_F(x) = \theta(x_0) \int \frac{d^3p}{(2\pi)^3} \frac{\not{p} + m}{2p_0} \theta(p_0 - \mu) e^{-ip \cdot x + i\mu x_0} - \theta(-x_0) \int \frac{d^3p}{(2\pi)^3} \left[ \frac{\not{p} + m}{2p_0} \theta(\mu - p_0) e^{-ip \cdot x + i\mu x_0} + \frac{\not{p} - m}{2p_0} e^{ip \cdot x + i\mu x_0} \right], \quad (4.2)$$

where  $p_0 = \sqrt{|\vec{p}|^2 + m^2}$ . We see that this propagator agrees with the one obtained by the canonical quantization [17] except the overall phase factor due to the shift in the energy by  $\mu$  to set the energy of the Fermi surface to be zero.

In terms of this full propagator Eq. (4.1), the quark-loop contribution to the one-loop vacuum polarization tensor becomes

$$\Pi_{ab\text{full}}^{\mu\nu}(p) = g_s^2 \int d^4x e^{-ip \cdot x} \langle J_a^\mu(x) J_b^\nu(0) \rangle \quad (4.3)$$

$$= \frac{iN_f}{2} g_s^2 \delta_{ab} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q_0} \frac{1}{2k_0} \left[ \frac{\theta(q_0 - \mu)\theta(\mu - k_0)}{q_0 - k_0 + p_0 - i\epsilon} T^{\mu\nu}(q, k) - \frac{\theta(\mu - q_0)\theta(k_0 - \mu)}{q_0 - k_0 + p_0 + i\epsilon} T^{\mu\nu}(q, k) \right. \\ \left. + \frac{\theta(q_0 - \mu)}{q_0 + k_0 + p_0 - i\epsilon} T^{\mu\nu}(q, k') - \frac{\theta(k_0 - \mu)}{-q_0 - k_0 + p_0 + i\epsilon} T^{\mu\nu}(q, k'') \right], \quad (4.4)$$

where  $T^{\mu\nu}(q, k) = \text{Tr}[\gamma^\mu \not{q} \gamma^\nu \not{k}]$ ,  $\vec{k} = \vec{q} + \vec{p}$ ,  $\vec{k}' = -\vec{q} - \vec{p}$ , and  $\vec{k}'' = -\vec{q} + \vec{p}$ .

If we rewrite the step functions as  $\theta(q_0 - \mu) = 1 - \theta(\mu - q_0)$  and  $\theta(k_0 - \mu) = 1 - \theta(\mu - k_0)$  for the last two terms in Eq. (4.4), one can easily see that the quark-loop contribution of the vacuum polarization tensor consists of two parts, one due to the matter and the other due to the vacuum,  $\Pi_{ab\text{full}}^{\mu\nu}(p) = \Pi_{ab\text{mat}}^{\mu\nu}(p) + \Pi_{ab\text{vac}}^{\mu\nu}(p)$ , where  $\Pi_{ab\text{vac}}^{\mu\nu}$  is the quark-loop contribution when there is no matter (*i.e.*  $\mu = 0$ ),

$$\Pi_{ab\text{vac}}^{\mu\nu} = \frac{iN_f}{2} g_s^2 \delta_{ab} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q_0} \frac{1}{2k_0} \left[ \frac{T^{\mu\nu}(q, k')}{q_0 + k_0 + p_0 - i\epsilon} - \frac{T^{\mu\nu}(q, k'')}{-q_0 - k_0 + p_0 + i\epsilon} \right], \quad (4.5)$$

where  $N_f$  is the number of light quark flavors. For  $|p^\mu| \ll \mu$ , the matter part of the vacuum polarization becomes, with  $M^2 = N_f g_s^2 \mu^2 / (2\pi^2)$ ,

$$\Pi_{ab\text{mat}}^{\mu\nu} = -\frac{iM^2}{2} \delta_{ab} \int \frac{d\Omega_{\vec{v}_F}}{4\pi} \left( \frac{-2\vec{p} \cdot \vec{v}_F V^\mu V^\nu}{p \cdot V + i\epsilon \vec{p} \cdot \vec{v}_F} + g^{\mu\nu} - \frac{V^\mu \bar{V}^\nu + \bar{V}^\mu V^\nu}{2} \right) \quad (4.6)$$

which is transversal,  $p_\mu \Pi_{ab\text{mat}}^{\mu\nu}(p) = 0$  [17].

Now, we calculate the quark contribution to the one-loop vacuum polarization tensor in the effective theory. Quarks in the (high density) effective theory are almost on-shell at low

energy and the quark current, which consists of states near the Fermi surface, is proportional to its velocity,  $V^\mu = (1, \vec{v}_F)$ :

$$J^{\mu a}(x) \equiv \sum_{\vec{v}_F} \bar{\psi}_+(\vec{v}_F, x) \gamma^\mu T^a \psi_+(\vec{v}_F, x) = \sum_{\vec{v}_F} V^\mu \bar{\psi}_+(\vec{v}_F, x) \gamma^0 T^a \psi_+(\vec{v}_F, x), \quad (4.7)$$

where we used  $P_+ \psi_+(\vec{v}_F, x) = \psi_+(\vec{v}_F, x)$  and  $P_- \gamma^\mu P_+ = P_- \gamma^0 V^\mu$ . Therefore, the gluons transversal to the quark velocity is decoupled at low energy and the gluons longitudinal to the quark velocity is coupled to the quark color charge density in a combination given as

$$A_\mu^a(x) J^{\mu a}(x) = \sum_{\vec{v}_F} \bar{\psi}_+(\vec{v}_F, x) \gamma^0 T^a \psi_+(\vec{v}_F, x) V \cdot A^a(x). \quad (4.8)$$

Therefore, in the leading order in  $1/\mu$  expansion, the quark-gluon coupling does not involve the spin of quarks like a scalar coupling. For a given quark, moving nearly at a Fermi momentum  $\mu \vec{v}_F$ , only one component of gluons, namely  $A_0 - \vec{v}_F \cdot \vec{A}$  combination, couples to the quark in the leading order.

With the quark current given in Eq. (4.7), we obtain the quark-loop contribution to the vacuum polarization in the effective theory as

$$\Pi_{ab}^{\mu\nu}(p) = g_s^2 \int d^4x e^{-ip \cdot x} \sum_{\vec{v}_F} \langle 0 | T J_a^\mu(\vec{v}_F, x) J_b^\nu(\vec{v}_F, 0) \rangle \quad (4.9)$$

$$\begin{aligned} &= -g_s^2 N_f \delta_{ab} \sum_{\vec{v}_F} V^\mu V^\nu \int \frac{d^4q}{(2\pi)^4} \text{Tr} \left[ \left( \frac{1 - \vec{\alpha} \cdot \vec{v}_F}{2} \right) \gamma^0 \frac{i}{\not{q}_\parallel + i\epsilon} \gamma^0 \frac{i}{\not{q}_\parallel + \not{p}_\parallel + i\epsilon} \right] \\ &= -\frac{i}{8\pi} \delta_{ab} \sum_{\vec{v}_F} V^\mu V^\nu \left[ 1 - \frac{(p^0 + \vec{p} \cdot \vec{v}_F)^2}{p_\parallel^2 + i\epsilon} \right] M^2, \end{aligned} \quad (4.10)$$

where  $p_\parallel^\mu = (p^0, \vec{v}_F \vec{v}_F \cdot \vec{p})$ . We note that the quark-loop contribution to the vacuum polarization tensor in the effective theory is exactly same as the sum of the first and second terms in the QCD vacuum polarization, Eq. (4.4). This is what it should be, since the effective theory does reproduce the contribution by quarks and holes in QCD when the external gluon momentum  $p \rightarrow 0$ . The third and fourth terms in Eq. (4.4), due to the quark and anti-quark pair creation, is absent in the effective theory because the anti-quarks are integrated out. The effect of anti-quarks will be taken into account when we match the gluon two-point amplitudes.

To match the gluon two-point amplitudes in both theories, we therefore need to add a term in the one-loop effective Lagrangian,

$$\mathcal{L}_{\text{eff}} \ni -\frac{M^2}{16\pi} \sum_{\vec{v}_F} A_\perp^{a\mu} A_{\perp\mu}^a, \quad (4.11)$$

which also ensures the gauge invariance of the effective Lagrangian at one-loop. The static screening mass can be read off from the vacuum polarization tensor in the limit  $p_0 \rightarrow 0$ , which is in the effective theory

$$\Pi_{ab}^{\mu\nu}(p_0 \rightarrow 0, \vec{p}) \simeq -iM^2 \delta_{ab} \delta^{\mu 0} \delta^{\nu 0}. \quad (4.12)$$

We see that the electric gluons,  $A_0^a$ , have a screening mass,  $M$ , but the static magnetic gluons are not screened at one-loop due to the added term Eq. (4.11), which holds at all orders in perturbation as in the finite temperature [24,15].

Finally, by matching the quark two-point amplitudes, we get a one-loop low-energy (Wilsonian) effective Lagrangian density;

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & -\frac{1}{4}(1+a_1)\left(F_{\mu\nu}^a\right)^2 - \frac{M^2}{16\pi}A_{\perp}^{a\mu}A_{\perp\mu}^a + (1+b_1)\bar{\psi}i\gamma_{\parallel}^{\mu}D_{\mu}\psi - \frac{1}{2\mu}(1+c_1)\psi^{\dagger}(\gamma_{\perp}\cdot D)^2\psi \\ & + \frac{1}{2\mu^2}\left[\left(g_3\delta_{us;tv}^A - g_6\delta_{us;tv}^S\right)\psi_t^{\dagger}(\vec{v}_F, x)\psi_s(\vec{v}_F, x)\psi_v^{\dagger}(-\vec{v}_F, x)\psi_u(-\vec{v}_F, x) \right. \\ & \left. + \left(h_3\delta_{us;tv}^A - h_6\delta_{us;tv}^S\right)\psi_t^{\dagger}(\vec{v}_F, x)\gamma_5\psi_s(\vec{v}_F, x)\psi_v^{\dagger}(-\vec{v}_F, x)\gamma_5\psi_u(-\vec{v}_F, x)\right] + \dots, \end{aligned} \quad (4.13)$$

where the summation over  $\vec{v}_F$  is suppressed and the coefficients  $a_1, b_1, c_1$  are dimensionless and of order  $\alpha_s(\mu)$ . The ellipsis denotes the irrelevant four-quark operators and terms with more external fields and derivatives.

## V. RG ANALYSIS AND GAP EQUATION

As we scale further down, the effective four-quark operators will evolve together with other operators, which can be seen by further integrating out the high frequency modes,  $s\mu < |l_{\mu}| < \mu$ . The scale dependence of the four-quark operators has three pieces. One is from the one-loop matching condition for the four-quark amplitudes and the other two are from the loop corrections to the four-quark operators. Putting all contribution together, we find the one-loop renormalization group equations for the four-quark operators [11] to be

$$s\frac{\partial}{\partial s}\bar{g}_i = -\gamma_i\alpha_s^2 - \frac{1}{4\pi^2}\bar{g}_i^2 - \frac{\ln 2}{12\pi}\delta_i\bar{g}_i\alpha_s, \quad (5.1)$$

where  $i = (6, \bar{3})$ ,  $\bar{g}_6 = -g_6$ ,  $\bar{g}_3 = g_3$  and  $\gamma_i = (\sqrt{2}/9)(13/4, 1/2)$  and  $\delta_i = (-1, 2)$ . Since in the high density limit the quark-gluon coupling is (1+1)-dimensional, the quarks do not contribute to the running of the strong coupling. The one-loop  $\beta$  function for the strong coupling constant at high density is  $\beta(\alpha_s) = -11/(2\pi)\alpha_s^2$ .

To solve the RG equation, Eq. (5.1), we introduce a new variable

$$y_i \equiv \frac{1}{22\pi\alpha_s(\mu)}\left(\bar{g}_i + \frac{\pi\ln 2}{6}\delta_i\alpha_s(\Lambda)\right) \quad \text{and} \quad t = \frac{11}{2\pi}\alpha_s(\mu)\ln s \quad (5.2)$$

Then, the RG equations becomes

$$\frac{dy_i}{dt} = -y_i^2 - \frac{a_i}{(1+t)^2}, \quad (5.3)$$

where  $a_i = (132\delta_i\ln 2 + 144\gamma_i - \delta_i^2)/(17424)$ . Further, letting  $y_i = f_i(t)/(1+t)$ , we get

$$(1+t)\frac{df_i}{dt} = -(f_i^2 - f_i + a_i). \quad (5.4)$$

Integrating Eq. (5.4), we get

$$f_i(t) = \frac{C_2(f_i(0) - C_1)(1+t)^{-(C_1-C_2)} - C_1(f_i(0) - C_2)}{(f_i(0) - C_1)(1+t)^{-(C_1-C_2)} - (f_i(0) - C_2)}, \quad (5.5)$$

where  $C_1 = (1 + \sqrt{1 - 4a_i})/2 = 1 - C_2$  and  $f_i(0) = (g_i(\mu) + \pi \ln 2\delta_i\alpha_s(\mu)/6)/(22\pi\alpha_s(\mu))$ . Since  $a_i \ll 1$ , we take  $C_1 \simeq 1$ ,  $C_2 \ll 1$ . And also, for  $\mu \gg \Lambda_{\text{QCD}}$ ,  $f_i(0) \simeq \delta_i \ln 2/(132) \ll 1$ . Therefore, for  $1+t \rightarrow 0^+$  or  $s \rightarrow \exp[-2\pi/(11\alpha_s(\mu))]$ , we get  $\bar{g}_i(t) + \pi \ln 2\delta_i\alpha_s(t)/6 \simeq 22\pi C_2\alpha_s(t)$ . Namely,

$$\bar{g}_i(\Lambda) \simeq \frac{2\pi}{11}\alpha_s(\Lambda) \left[ \gamma_i - \frac{(\ln 2)^2}{144}\delta_i^2 \right]. \quad (5.6)$$

At a scale much less than the chemical potential,  $\Lambda \ll \mu$ ,  $g_6(\Lambda) \simeq 0.29\alpha_s(\Lambda)$  and  $g_3(\Lambda) \simeq 0.04\alpha_s(\Lambda)$ . Similarly for  $h_i$ ,  $\gamma_i = (\sqrt{2}/2)(2, 1)$  and  $\delta_i = (-1, 2)$  and we get  $h_6(\Lambda) = 0.81\alpha_s(\Lambda)$  and  $h_3(\Lambda) \simeq 0.4\alpha_s(\Lambda)$ .

At a scale below the screening mass, we further integrate out the electric gluons, which will generate four-quark interactions. For quarks moving with opposite Fermi momenta, the electric-gluon exchange four-quark interaction is given as

$$\mathcal{L}_{1g} \ni -\frac{g_s^2(M)}{2M^2} \sum_{\vec{v}_F} \bar{\psi}\gamma^0 T_a \psi(\vec{v}_F, x) \bar{\psi}\gamma_0 T_a \psi(-\vec{v}_F, x). \quad (5.7)$$

Using  $T_{tu}^a T_{vs}^a = 1/2\delta_{ts}\delta_{uv} - 1/6\delta_{tu}\delta_{vs}$ , we find that the four-quark couplings are shifted as  $g_3(M) \rightarrow g_3(M) + 2\sqrt{2}g_s^2(M)/3 \simeq 0.95g_s^2(M)$  and  $g_6(M) \rightarrow g_6(M) + \sqrt{2}g_s^2(M)/3 \simeq 0.49g_s^2(M)$ , while  $h_i$ 's are unchanged since Eq. (5.7) does not involve  $\gamma_5$ .

As we approach further to the Fermi surface, closer than the screening mass  $M$ , the four-quark operators in the color anti-triplet channel become stronger, because the  $\beta$  function for the attractive four-quark operators is negative,  $\beta(g_3) = -g_3^2/(4\pi^2)$ . If the four-quark interaction is dominant at low energy, it leads to vacuum instability in the infrared region by forming a color anti-triplet condensate or Cooper pair. But, since the long-range color-magnetic interactions also become strong at low energy, we need to consider both interactions to determine the Cooper-pair gap.

Since both relevant interactions are attractive for a pair of quarks in color anti-triplet channel with opposite Fermi momenta, they may lead to condensates of quark pairs in color anti-triplet channel. To describe the Cooper-pair gap equation, we introduce a charge conjugate field,

$$(\psi_c)_i(\vec{v}_F, x) = C_{ij}\bar{\psi}_j(-\vec{v}_F, x), \quad (5.8)$$

where  $i$  and  $j$  are Dirac indices and the matrix  $C$  satisfies  $C^{-1}\gamma_\mu C = -\gamma_\mu^T$ . Then, we can write the inverse propagator for  $\Psi(\vec{v}_F, x) \equiv (\psi(\vec{v}_F, x), \psi_c(\vec{v}_F, x))^T$  as

$$S^{-1}(\vec{v}_F, l) = \gamma_0 \begin{pmatrix} Z(l_\parallel)l \cdot V & -\Delta(l_\parallel) \\ -\Delta^\dagger(l_\parallel) & Z(l_\parallel)l \cdot \bar{V} \end{pmatrix}, \quad (5.9)$$

where  $Z(l_\parallel)$  is the wave function renormalization constant and  $\Delta(l_\parallel)$  is the Cooper-pair gap. The Cooper-pair gap is nothing but a Majorana mass for quarks. When Cooper-pairs form and Bose-condense, the quarks get a Majorana mass dynamically and the Fermi sea opens up a gap.

Schwinger-Dyson (SD) equations are infinitely coupled integral equations for Green functions. In order to solve the Schwinger-Dyson equations, we need to truncate them consistently. Since  $g_s(M)$  is still weak at high density  $M \gg \Lambda_{\text{QCD}}$ , one may use the ladder approximation which is basically the weak coupling expansion of the SD equations, consistently with the Ward-Takahashi identity. Another gauge-invariant truncation is so-called hard-dense-loop (HDL) re-summation, which can be derived gauge invariantly from the transport equation [17] as in the finite temperature [25]. When the gluon momentum is of order of the screening mass  $M \sim g_s \mu$ , the quark one-loop vacuum polarization is also same order as we can see from Eq. (4.10). Therefore, if the important momentum scale of the diagrams is of order of  $M$ , one needs to re-sum all the bubble diagrams to get the consistent gluon propagator. Since the HDL re-summed gluon propagator correctly incorporates the medium effects like screening at high density, we will solve the SD equations for the quark propagator in the HDL approximation to calculate the Cooper-pair gap.

In the HDL approximation, the SD equations for the quark propagator are given as, neglecting small  $h_i$  four-Fermi couplings,

$$[Z(p_{\parallel}) - 1] p \cdot V = (-ig_s)^2 \int_l V^{\mu} D_{\mu\nu}(p-l) \bar{V}^{\nu} \frac{T^a Z(l_{\parallel}) l \cdot V T^{aT}}{Z^2 l_{\parallel}^2 - \Delta(l_{\parallel})^2} + \frac{g_3}{\mu^2} \int_l \frac{Z(l_{\parallel}) l \cdot V}{Z^2 l_{\parallel}^2 - \Delta(l_{\parallel})^2} \quad (5.10)$$

$$\Delta(p_{\parallel}) = (-ig_s)^2 \int_l V^{\mu} D_{\mu\nu}(p-l) \bar{V}^{\nu} \frac{T^a \Delta(l_{\parallel}) T^{aT}}{Z^2 l_{\parallel}^2 - \Delta(l_{\parallel})^2} + \frac{g_3}{\mu^2} \int_l \frac{\Delta(l_{\parallel})}{Z^2 l_{\parallel}^2 - \Delta(l_{\parallel})^2}, \quad (5.11)$$

where the gluon propagator  $D_{\mu\nu}$  is given in the HDL approximation as,<sup>3</sup> following the notations used by Schäfer and Wilczek [13],

$$iD_{\mu\nu}(k) = \frac{P_{\mu\nu}^T}{k^2 - G} + \frac{P_{\mu\nu}^L}{k^2 - F} - \xi \frac{k_{\mu} k_{\nu}}{k^4}, \quad (5.12)$$

where  $\xi$  is the gauge parameter and the projectors are defined by

$$P_{ij}^T = \delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}, \quad P_{00}^T = 0 = P_{0i}^T \quad (5.13)$$

$$P_{\mu\nu}^L = -g_{\mu\nu} + \frac{k_{\mu} k_{\nu}}{k^2} - P_{\mu\nu}^T. \quad (5.14)$$

Before solving the SD equations, we first show that the preferred solution of the SD equations supports the color-flavor locking condensate predicted in [7]. Since the gluon interaction is vectorial, the gluon exchange interaction in the gap equation does not distinguish the handedness of quarks and thus it will generate same condensates regardless of handedness;  $|\langle \psi_L \psi_L \rangle| = |\langle \psi_R \psi_R \rangle| = |\langle \psi_L \psi_R \rangle|$ , suppressing other quantum numbers. But, the four-Fermi interaction in the effective Lagrangian, Eq. (4.13), is chiral, which can be seen if we rewrite it in the handedness basis as following;

---

<sup>3</sup>In the Schwinger-Dyson equation the loop momentum should take the whole range up to the ultraviolet cutoff, which is the chemical potential  $\mu$  in the case of high density effective theory. Hence the gluon propagator includes both magnetic and electric gluons.

$$\begin{aligned} \mathcal{L}_{4f}^1 \ni & \frac{g_3 + h_3}{2\mu^2} \delta_{us;tv}^A \left[ \psi_{Lt}^\dagger(\vec{v}_F, x) \psi_{Ls}(\vec{v}_F, x) \psi_{Lv}^\dagger(-\vec{v}_F, x) \psi_{Lu}(-\vec{v}_F, x) + (L \leftrightarrow R) \right] \\ & + \frac{g_3 - h_3}{2\mu^2} \delta_{us;tv}^A \left[ \psi_{Lt}^\dagger(\vec{v}_F, x) \psi_{Ls}(\vec{v}_F, x) \psi_{Rv}^\dagger(-\vec{v}_F, x) \psi_{Ru}(-\vec{v}_F, x) + (L \leftrightarrow R) \right]. \end{aligned} \quad (5.15)$$

We see that the  $LL$  (or  $RR$ ) four-Fermi coupling is bigger than the  $LR$  four-Fermi coupling, since  $h_3 > 0$ . Therefore, the gap in  $LL$  or  $RR$  channel will be bigger than the one in  $LR$  channel due to the difference in the four-Fermi couplings. Thus, the  $LL$  or  $RR$  condensate is energetically more preferred than the  $LR$  condensate. We also note that in the effective theory the gluons are blind not only to flavors but also to the Dirac indices of quarks because they couple to quark currents in the combination of  $V \cdot A$  like a scalar field, as given in Eq. (4.8). Therefore, in the SD equations, the diquark Cooper-pair can be decomposed into color anti-triplet and color sextet. But, since quarks of opposite momenta are attractive in the anti-triplet channel while repulsive in the sextet channel, without loss of generality, we can write the Cooper-pair gap in color anti-triplet <sup>4</sup>. Since quarks are anti-commuting, the only possible way to form diquark (S-wave) condensate is either in spin-singlet or in spin-triplet:

$$\langle \psi_{Li\alpha}^a(\vec{v}_F, x) \psi_{Lj\beta}^b(-\vec{v}_F, x) \rangle = - \langle \psi_{Ri\alpha}^a(\vec{v}_F, x) \psi_{Rj\beta}^b(-\vec{v}_F, x) \rangle \quad (5.16)$$

$$= \epsilon_{ij} \epsilon^{abc} K_{[\alpha\beta]c}(p_F) + \delta_{ij} \epsilon^{abc} K_{\{\alpha\beta\}c}(p_F), \quad (5.17)$$

where  $a, b, c = 1, 2, 3$  are color indices,  $\alpha, \beta, \gamma = u, d, s, \dots, N_f$  flavor indices, and  $i, j = 1, 2$  spinor indices. Indices in the bracket and in the curled bracket are anti-symmetrized and symmetrized, respectively. But, the spin-one component of the gap,  $K_{\{\alpha\beta\}c}$ , vanishes algebraically, since  $\psi(\vec{v}_F, x) = 1/2 (1 + \vec{\alpha} \cdot \vec{v}_F) \psi(\vec{v}_F, x)$  and  $(1 + \vec{\alpha} \cdot \vec{v}_F)_{il} (1 - \vec{\alpha} \cdot \vec{v}_F)_{lj} = 0$ .

When  $N_f = 3$ , the spin-zero component of the condensate becomes (flavor) anti-triplet,

$$K_{[\alpha\beta]c}(p_F) = \epsilon_{\alpha\beta\gamma} K_c^\gamma(p_F). \quad (5.18)$$

Using the global color and flavor symmetry, one can always diagonalize the spin-zero condensate as  $K_c^\gamma = \delta_c^\gamma K_\gamma$ . To determine the parameters,  $K_u$ ,  $K_d$ , and  $K_s$ , we need to minimize the vacuum energy for the condensate. By the Cornwall-Jackiw-Tomboulis formalism [27], the vacuum energy in the HDL approximation is given as

$$\begin{aligned} V(\Delta) &= -\text{Tr} \ln S^{-1} + \text{Tr} \ln \not{\partial} + \text{Tr} (S^{-1} - \not{\partial}) S + (2\text{PI diagrams}) \\ &= \frac{\mu^2}{4\pi} \sum_{i=1}^9 \int \frac{d^2 l_\parallel}{(2\pi)^2} \left[ \ln \left( \frac{l_\parallel^2}{l_\parallel^2 + \Delta_i^2(l_\parallel)} \right) + \frac{1}{2} \cdot \frac{\Delta_i^2(l_\parallel)}{l_\parallel^2 + \Delta_i^2(l_\parallel)} \right] + h.o., \end{aligned} \quad (5.19)$$

where  $h.o.$  are the higher order terms in the HDL approximation, containing more powers of coupling  $g_s$ , and  $\Delta_i$ 's are the eigenvalues of color anti-symmetric and flavor anti-symmetric  $9 \times 9$  gap,  $\Delta_{\alpha\beta}^{ab}$ . The 2PI diagrams are two-particle-irreducible vacuum diagrams. There is only one such diagram (see Fig. 3) in the leading order HDL approximation.

---

<sup>4</sup> At high but finite density, the Cooper-pair gap contains a small component of color-sextet [26]. But we will ignore this for simplicity.

Since the gap depends only on energy in the leading order, one can easily perform the momentum integration in (5.19) to get<sup>5</sup>,

$$\begin{aligned} V(\Delta) &= \frac{\mu^2}{4\pi^2} \int_0^\infty dl_0 \left( -\frac{\Delta_i^2}{\sqrt{l_0^2} + \sqrt{l_0^2 + \Delta_i^2}} + \frac{1}{4} \cdot \frac{\Delta_i^2}{\sqrt{l_0^2 + \Delta_i^2}} \right) \\ &\simeq -0.43 \frac{\mu^2}{4\pi^2} \sum_i |\Delta_i(0)|^2, \end{aligned} \quad (5.20)$$

where in the second line we used an approximation that

$$\Delta_i(l_0) \simeq \begin{cases} \Delta_i(0) & \text{if } |l_0| < |\Delta_i(0)|, \\ 0 & \text{otherwise.} \end{cases} \quad (5.21)$$

Were  $\Delta_i$  independent of each other, the global minimum should occur at  $\Delta_i(0) = \text{const.}$  for all  $i = 1, \dots, 9$ . But, due to the global color and flavor symmetry, only three of them are independent. Similarly to the condensate, the gap can be also diagonalized by the color and flavor symmetry as

$$\Delta_{ab}^{\alpha\beta} = \epsilon_{\alpha\beta\gamma} \epsilon^{abc} \Delta_\gamma \delta_c^\gamma. \quad (5.22)$$

Without loss of generality, we can take  $|\Delta_u| \geq |\Delta_d| \geq |\Delta_s|$ . Let  $\Delta_d/\Delta_u = x$  and  $\Delta_s/\Delta_u = y$ . Then, the vacuum energy becomes

$$V(\Delta) \simeq -0.43 \frac{\mu^2}{4\pi^2} |\Delta_u|^2 f(x, y), \quad (5.23)$$

where  $f(x, y)$  is a complicate function of  $-1 \leq x, y \leq 1$  that has a maximum at  $x = 1 = y$ ,  $f(x, y) \leq 13.4$ . Therefore, the vacuum energy has a global minimum when  $\Delta_u = \Delta_d = \Delta_s$ , or in terms of the eigenvalues of the gap

$$\Delta_i = \Delta_u \cdot (1, 1, 1, -1, 1, -1, 1, -1, -2). \quad (5.24)$$

Among nine quarks,  $\psi_a^\alpha$ , eight have (Majorana) mass  $\Delta_u$  and one has mass  $2\Delta_u$ .

Since the condensate is related to the off-diagonal component of the quark propagator at high momentum as, suppressing the color and flavor indices,

$$\begin{aligned} \langle \psi(\vec{v}_F, x) \psi(-\vec{v}_F, x) \rangle &\sim \lim_{y \rightarrow x} \int \frac{d^4 l}{(2\pi)^4} e^{il \cdot (x-y)} \frac{\Delta(l_\parallel)}{l_\parallel^2 - \Delta^2(l_\parallel)} \\ &= \lim_{y \rightarrow x} \left[ \delta^2(\vec{x}_\perp - \vec{y}_\perp) \frac{\Delta(0)}{4\pi^2 |x_\parallel - y_\parallel|^{\gamma_m}} + \dots \right], \end{aligned} \quad (5.25)$$

where  $\gamma_m$  is the anomalous dimension of the condensate and the ellipsis are less singular terms. Being proportional to the gap, the condensate is diagonalized in the basis where the

---

<sup>5</sup> If the condensate forms, the vacuum energy due to the gluons also depends on the gap due to the Meisner effect. But, it turns out to be subleading, compared to the quark vacuum energy;  $V_g(\Delta) \sim M^2 \Delta^2 \ln(\Delta/\mu) \sim g_s \mu^2 \Delta^2$  [28].

gap is diagonalized. Thus, we have shown that in the HDL approximation the true ground state of QCD with three massless flavors at high density is the color-flavor locking phase,  $K_\gamma = K$  for all  $\gamma = u, d, s$ . The condensate takes

$$\langle \psi_{L i \alpha}^a(\vec{v}_F, x) \psi_{L j \beta}^b(-\vec{v}_F, x) \rangle = - \langle \psi_{R i \alpha}^a(\vec{v}_F, x) \psi_{R j \beta}^b(-\vec{v}_F, x) \rangle = \epsilon_{ij} \epsilon^{abI} \epsilon_{\alpha\beta I} K(p_F), \quad (5.26)$$

breaking the color symmetry,  $U(1)_{\text{em}}$ , the chiral symmetry, and the baryon number symmetry. The symmetry breaking pattern of the CFL phase is therefore

$$SU(3)_c \times SU(3)_L \times SU(3)_R \times U(1)_{\text{em}} \times U(1)_B \mapsto SU(3)_V \times U(1)_{\tilde{Q}} \times Z_2, \quad (5.27)$$

where  $SU(3)_V$  is the diagonal subgroup of three  $SU(3)$  groups and the generator of  $U(1)_{\tilde{Q}}$  is a linear combination of the color hypercharge and  $U(1)_{\text{em}}$  generator,

$$\tilde{Q} = \cos \theta Q_{\text{em}} + \sin \theta Y_8, \quad (5.28)$$

where  $\tan \theta = e/g_s$ .

Now, we analyze the SD gap equation to see if it admits a nontrivial solution. Since the color-flavor locking gap is preferred if it exists, we may write the gap as

$$\Delta_{\alpha\beta}^{ab} = \epsilon^{abI} \epsilon_{\alpha\beta I} \Delta. \quad (5.29)$$

Then the gap equation becomes, neglecting small  $h_i$  couplings,

$$\Delta(p_{\parallel}) = (-ig_s)^2 \int \frac{d^4 l}{(2\pi)^4} D_{\mu\nu}(p-l) V^\mu \frac{T^a \Delta(l_{\parallel})(T^a)^T}{l_{\parallel}^2 - \Delta^2(l_{\parallel})} \bar{V}^\nu + i \frac{g_3}{\mu^2} \int \frac{d^4 l}{(2\pi)^4} \frac{\Delta(l_{\parallel})}{l_{\parallel}^2 - \Delta^2(l_{\parallel})}, \quad (5.30)$$

where we use the bare vertex and take  $Z(p_{\parallel}) = 1$  in the leading HDL approximation.

In the weak coupling limit,  $|k_0| \ll |\vec{k}|$  and thus

$$F(k_0, \vec{k}) \simeq M^2, \quad G(k_0, \vec{k}) \simeq \frac{\pi}{4} M^2 \frac{k_0}{|\vec{k}|}. \quad (5.31)$$

Since the gap has to be fully antisymmetric in color indices, we get

$$T_{tu}^a \Delta_{uv} (T^a)_{vs}^T = \left( \frac{1}{2} \delta_{tv} \delta_{us} - \frac{1}{6} \delta_{tu} \delta_{vs} \right) \Delta_{uv} = -\frac{2}{3} \Delta_{ts} \quad (5.32)$$

After Wick-rotating into Euclidean space, the gap equation becomes

$$\begin{aligned} \Delta(p_{\parallel}) = & \int \frac{d^4 q}{(2\pi)^4} \left[ -\frac{2}{3} g_s^2 \left\{ \frac{V \cdot P^T \cdot \bar{V}}{(p-q)_{\parallel}^2 + \vec{q}_{\perp}^2 + \frac{\pi}{4} M^2 |p_0 - q_0| / |\vec{p} - \vec{q}|} \right. \right. \\ & \left. \left. - \frac{1}{(p-q)_{\parallel}^2 + \vec{q}_{\perp}^2 + M^2} - \xi \frac{(p-q)_{\parallel}^2}{(p-q)^4} \right\} + \frac{g_3}{\mu^2} \right] \frac{\Delta(q_{\parallel})}{q_{\parallel}^2 + \Delta^2(q_{\parallel})}. \end{aligned} \quad (5.33)$$

Note that the main contribution to the integration comes from the loop momenta in the region  $q_{\parallel}^2 \sim \Delta^2$  and  $|\vec{q}_{\perp}| \sim M^{2/3} \Delta^{1/3}$ . Therefore, we find that the leading contribution is by

the first term due to the Landau-damped magnetic gluons. For this momentum range, we can take  $|\vec{p} - \vec{q}| \sim |\vec{q}_\perp|$  and

$$V \cdot P^T \cdot \bar{V} = -v_F^i v_F^j (\delta_{ij} - \hat{k}_i \hat{k}_j) = -1 + O\left(\frac{\Delta^{4/3}}{M^{4/3}}\right). \quad (5.34)$$

We also note that the term due to the four-Fermi operator is negligible, since  $g_3 \sim g_s^4$  at the matching scale  $\mu$ .

Neglecting  $(p - q)_\parallel^2$  in the denominator, the gap equation becomes at the leading order in the weak coupling expansion and  $1/\mu$  expansion

$$\Delta(p_\parallel) = \frac{2g_s^2}{3} \int \frac{d^4 q}{(2\pi)^4} \left[ \frac{1}{\vec{q}_\perp^2 + \frac{\pi}{4} M^2 |p_0 - q_0| |\vec{q}_\perp|} + \frac{1}{\vec{q}_\perp^2 + M^2} + \xi \frac{(p - q)_\parallel^2}{|\vec{q}_\perp|^4} \right] \frac{\Delta(q_\parallel)}{q_\parallel^2 + \Delta^2(q_\parallel)}. \quad (5.35)$$

The  $\vec{q}_\perp$  integration can now be performed easily to get

$$\Delta(p_\parallel) = \frac{g_s^2}{9\pi} \int \frac{d^2 q_\parallel}{(2\pi)^2} \frac{\Delta(q_\parallel)}{q_\parallel^2 + \Delta^2} \left[ \ln \left( \frac{\mu^3}{\frac{\pi}{4} M^2 |p_0 - q_0|} \right) + \frac{3}{2} \ln \left( \frac{\mu^2}{M^2} \right) + \frac{3}{2} \xi \right]. \quad (5.36)$$

We see that in this approximation  $\Delta(p_\parallel) \simeq \Delta(p_0)$ . Then, we can integrate over  $\vec{v}_F \cdot \vec{q}$  to get

$$\Delta(p_0) = \frac{g_s^2}{36\pi^2} \int_{-\mu}^{\mu} dq_0 \frac{\Delta(q_0)}{\sqrt{q_0^2 + \Delta^2}} \ln \left( \frac{\bar{\Lambda}}{|p_0 - q_0|} \right) \quad (5.37)$$

where  $\bar{\Lambda} = 4\mu/\pi \cdot (\mu/M)^5 e^{3/2\xi}$ . If we take  $\Delta \simeq \Delta(0)$  for a rough estimate of the gap,

$$1 = \frac{g_s^2}{36\pi^2} \left[ \ln \left( \frac{\bar{\Lambda}}{\Delta} \right) \right]^2 \quad \text{or} \quad \Delta \simeq \bar{\Lambda} \exp \left( -\frac{6\pi}{g_s} \right). \quad (5.38)$$

As was done by Son [10], one can convert the Schwinger-Dyson gap equation (5.37) into a differential equation to take into account the energy dependence of the gap. Approximating the logarithm in the gap equation as

$$\ln |p_0 - q_0| \simeq \begin{cases} \ln |p_0| & \text{if } |p_0| > |q_0|, \\ \ln |q_0| & \text{otherwise,} \end{cases} \quad (5.39)$$

we get

$$p\Delta''(p) + \Delta'(p) + \frac{2\alpha_s}{9\pi} \frac{\Delta(p)}{\sqrt{p^2 + \Delta^2}} = 0, \quad (5.40)$$

with boundary conditions  $p\Delta' = 0$  at  $p = \Delta$  and  $\Delta = 0$  at  $p = \bar{\Delta}$ , where  $p \equiv p_0$ . When  $p \ll \Delta(p)$ , the equation becomes

$$p\Delta'' + \Delta' + \frac{r^2}{4} \frac{\Delta(p)}{|\Delta|} = 0, \quad (5.41)$$

where  $r^2 = 2g_s^2/(9\pi^2)$  and  $|\Delta|$  is the gap at  $p = 0$ . We find  $\Delta(p) = |\Delta| J_0 \left( r\sqrt{p/|\Delta|} \right)$  for  $p \ll |\Delta|$ . When  $p \gg \Delta$ , the differential equation (5.40) becomes

$$p\Delta'' + \Delta' + \frac{r^2}{4} \frac{\Delta}{p} = 0, \quad (5.42)$$

which has a power solution,  $\Delta \sim p^{\pm ir/2}$ . Since the gap vanishes at  $p = \bar{\Lambda}$ , we get for  $p \gg \Delta$

$$\Delta(p) = B \sin \left( \frac{r}{2} \ln \frac{\bar{\Lambda}}{p} \right). \quad (5.43)$$

By matching two solutions at the boundary  $p = |\Delta|$  we get

$$B \simeq |\Delta| \quad \text{and} \quad |\Delta| = \bar{\Lambda} e^{-\pi/r}. \quad (5.44)$$

The gap is therefore given as at the leading order in the weak coupling expansion<sup>6</sup>

$$|\Delta| = c \cdot \frac{\mu}{g_s^5} \exp \left( -\frac{3\pi^2}{\sqrt{2}g_s} \right), \quad (5.45)$$

where  $c = 2^7 \pi^4 N_f^{-5/2} e^{3\xi/2+1}$ . This agrees with the RG analysis done by Son [10] (see also [29]) and also with the Schwinger-Dyson approach in full QCD [12–14]. The  $1/g_s$  behavior of the exponent of the gap at high density is due to the double logarithmic divergence in the gap equation (5.33), similarly to the case of chiral symmetry breaking under external magnetic fields [22,30,31]. In addition to the usual logarithmic divergence in the quark propagator as in the BCS superconductivity, there is another logarithmic divergence due to the long-range gluon exchange interaction, which occurs when the gluon loop momentum is colinear to the incoming quark momentum ( $\vec{q}_\perp \rightarrow 0$ ).

## VI. TEMPERATURE EFFECTS AND HIGHER ORDER CORRECTIONS

When the quark matter is not very dense and not very cold, the effects of finite temperature and density become important. In this section we calculate the critical density and temperature. First, we add the  $1/\mu$  corrections to the gap equation Eq. (5.30) to see how the formation of Cooper pair changes when the density decreases. As derived in [11], the leading  $1/\mu$  corrections to the quark-gluon interactions are

$$\mathcal{L}_1 = -\frac{1}{2\mu} \sum_{\vec{v}_F} \psi^\dagger(\vec{v}_F, x) (\gamma_\perp \cdot D)^2 \psi(\vec{v}_F, x) = -\sum_{\vec{v}_F} \left[ \psi^\dagger \frac{D_\perp^2}{2\mu} \psi + g_s \psi^\dagger \frac{\sigma_{\mu\nu} F^{\mu\nu}}{4\mu} \psi \right]. \quad (6.1)$$

In the leading order in the HDL approximation, the loop correction to the vertex is neglected and the quark-gluon vertex is shifted by the  $1/\mu$  correction as

$$-ig_s v_F^i \mapsto -ig_s v_F^i - ig_s \frac{l_\perp^i}{\mu}, \quad (6.2)$$

---

<sup>6</sup> The gauge-parameter dependent term is subleading in the gap equation (5.33). Since the gap has to be gauge-independent, the gauge parameter dependence in the prefactor will disappear if one includes the higher order corrections.

where  $l_i$  is the momentum carried away from quarks by gluons. Then, the gap equation (5.35) becomes

$$\Delta(p_{\parallel}) = \frac{2g_s^2}{3} \int \frac{d^4l}{(2\pi)^4} \left[ \frac{|\vec{l}_{\perp}| (1 - l_{\perp}^2/\mu^2)}{|\vec{l}_{\perp}|^3 + (\pi/4)M^2|l_0 - p_0|} + \frac{1 - l_{\perp}^2/\mu^2}{l_{\perp}^2 + M^2} + \xi \cdot \frac{l_{\parallel}^2 (1 - l_{\perp}^2/\mu^2)}{l^4} \right] \frac{\Delta(l_{\parallel})}{l_{\parallel}^2 + \Delta^2}. \quad (6.3)$$

For a constant gap approximation,  $\Delta(p_{\parallel}) \simeq \Delta$ , the gap equation becomes in the leading order, as  $p \rightarrow 0$ ,

$$\begin{aligned} 1 &= \frac{g_s^2}{9\pi} \int \frac{d^2l_{\parallel}}{(2\pi)^2} \left[ \ln \left( \frac{\bar{\Lambda}}{|l_0|} \right) - \frac{3}{2} \right] \frac{1}{l_{\parallel}^2 + \Delta^2} \\ &= \frac{g_s^2}{36\pi^2} \ln \left( \frac{\bar{\Lambda}}{\Delta} \right) \left[ \ln \left( \frac{\bar{\Lambda}}{\Delta} \right) - 3 \right]. \end{aligned} \quad (6.4)$$

Therefore, we see that, when  $\mu < \mu_c \simeq e^3 \Delta$ , the gap due to the long-range color magnetic interaction disappears. Since the phase transition for color superconducting phase is believed to be of first order [32,33], we may assume that the gap has the same dependence on the chemical potential  $\mu$  as the leading order. Then, the critical density for the color superconducting phase transition is given by

$$\mu_c = e^3 \mu_c \exp \left[ -\frac{4\pi\sqrt{3}}{g_s(\mu_c)} \right]. \quad (6.5)$$

Therefore, if the strong interaction coupling is too strong at the scale of the chemical potential, the gap does not form. To form the Cooper pair gap, the strong coupling at the scale of the chemical potential has to be smaller than  $g_s(\mu_c) = \pi^2/\sqrt{2}$ . By using the one-loop  $\beta$  function for three light flavors,  $\beta(g_s) = -9/(16\pi^2)g_s^3$ , and the experimental value for the strong coupling constant,  $\alpha_s(1.73\text{GeV}) = 0.32_{-0.053}^{+0.031}(\text{exp}) \pm 0.016(\text{theo})$  [34], we get  $0.13\text{GeV} \lesssim \mu_c \lesssim 0.31\text{GeV}$ , which is about the same order as the one estimated by the instanton induced four-Fermi interaction [33,35] or by general effective four-Fermi interactions [32]. But, this should be taken as an order of magnitude, since for such a small chemical potential the higher order terms in  $1/\mu$  expansion, which we have neglected, are as important as the leading term.

So far we have not included the temperature effect in analyzing the gap. The temperature effect is quite important to understand the heavy ion collision or the final stage of the evolution of giant stars because quarks will have to have high energy to bring together to form a super dense matter. The super dense and hot quark matter will go through a phase transition as it cools down by emitting weakly interacting particles like neutrinos.

At finite temperature,  $T$ , the gap equation (5.35) becomes, following the imaginary-time formalism developed by Matsubara [36],

$$\Delta(\omega_{n'}) = \frac{g_s^2}{9\pi} T \sum_{n=-\infty}^{+\infty} \int \frac{dq}{2\pi} \frac{\Delta(\omega_n)}{\omega_n^2 + \Delta^2(\omega_n) + q^2} \ln \left( \frac{\bar{\Lambda}}{|\omega_{n'} - \omega_n|} \right), \quad (6.6)$$

where  $\omega_n = \pi T(2n+1)$  and  $q \equiv \vec{v}_F \cdot \vec{q}$ . We now use the constant (but temperature-dependent) gap approximation,  $\Delta(\omega_n) \simeq \Delta(T)$  for all  $n$ . Taking  $n' = 0$  and converting the logarithm into integration, we get

$$\Delta(T) = \frac{g_s^2}{18\pi} T \sum_{n=-\infty}^{+\infty} \int \frac{dq}{2\pi} \int_0^{\bar{\Lambda}^2} dx \frac{\Delta(T)}{\omega_n^2 + \Delta^2(T) + q^2} \cdot \frac{1}{x + (\omega_n - \omega_0)^2}. \quad (6.7)$$

Using the contour integral (see Fig. 4) [37], one can in fact sum up over all  $n$  to get

$$1 = \frac{g_s^2}{36\pi^2} T \int dq \int_0^{\bar{\Lambda}^2} dx \frac{1}{2\pi i} \oint_C \frac{d\omega}{1 + e^{-\omega/T}} \cdot \frac{1}{(\omega^2 - q^2 - \Delta^2(T))[(\omega_n - i\omega_0)^2 + x]}. \quad (6.8)$$

Since the gap vanishes at the critical temperature,  $\Delta(T_C) = 0$ , after performing the contour integration in Eq. (6.8), we get

$$1 = \frac{g_s^2}{36\pi^2} \int dq \int_0^{\bar{\Lambda}^2} dx \left\{ \frac{(\pi T_C)^2 + x - q^2}{[(\pi T_C)^2 + x - q^2]^2 + (2\pi T_C q)^2} \cdot \frac{\tanh[q/(2T_C)]}{2q} \right. \\ \left. + \frac{(\pi T_C)^2 + q^2 - x}{[(\pi T_C)^2 + q^2 - x]^2 + (2\pi T_C)^2 x} \cdot \frac{\coth[\sqrt{x}/(2T_C)]}{\sqrt{2}} \right\}. \quad (6.9)$$

At high density  $\bar{\Lambda} \gg T_C$ , the second term in the integral in Eq. (6.9) is negligible, compared to the first term, and integrating over  $x$ , we get

$$1 = \frac{g_s^2}{36\pi^2} \int_0^{\lambda_c} dy \frac{\tanh y}{y} \left[ \ln \left( \frac{\lambda_c^2}{(\pi/2)^2 + y^2} \right) + O \left( \frac{y^2}{\lambda_c^2} \right) \right] \\ = \frac{g_s^2}{36\pi^2} \left[ \int_0^1 dy \frac{\tanh y}{y} \ln \lambda_c^2 + \int_1^{\lambda_c} dy \frac{\tanh y}{y} \ln \frac{\lambda_c^2}{y^2} + \dots \right] \\ = \frac{g_s^2}{36\pi^2} [(\ln \lambda_c)^2 + 2A \ln \lambda_c + \text{const.}] \quad (6.10)$$

where we have introduced  $y \equiv q/(2T_C)$  and  $\lambda_c \equiv \bar{\Lambda}/(2T_C)$  and  $A$  is given as

$$A = \int_0^1 dy \frac{\tanh y}{y} + \int_1^\infty dy \frac{\tanh y - 1}{y} = \ln \left( \frac{4}{\pi} \right) + \gamma. \quad (6.11)$$

Therefore, we find, taking the Euler-Mascheroni constant  $\gamma \simeq 0.577$ ,

$$T_C = \frac{e^A}{2} \Delta \simeq 1.13 \Delta. \quad (6.12)$$

As comparison, we note in the BCS case, which has a contact four-Fermi interaction with strength  $\bar{g}$ , the critical temperature is given as

$$1 = \bar{g} \int_0^{\tilde{\omega}_c} dz \frac{\tanh z}{z} \\ \simeq \bar{g} \left[ \int_1^{\tilde{\omega}_c} \frac{dz}{z} + \int_0^1 dz \frac{\tanh z}{z} - \int_1^\infty dz \frac{1 - \tanh z}{z} \right] \\ = \bar{g} \ln(e^A \tilde{\omega}_c) \quad (6.13)$$

where  $\tilde{\omega}_c (\gg 1)$  is determined by the Debye energy,  $\tilde{\omega}_c = \omega_D/(2T_C)$ . It shows that the ratio between the critical temperature and the Cooper-pair gap<sup>7</sup> in color superconductivity is same as the BCS value,  $e^\gamma/\pi \simeq 0.57$  [14,38,39].

---

<sup>7</sup>In the literature, the BCS gap is defined as twice of the dynamical mass,  $2\Delta$  [42].

## VII. APPLICATIONS

It is believed that the core of compact stars like neutron stars may be dense enough to form quark matter and may shed some lights on understanding QCD at high density. The properties of compact stars can be investigated by studying the emission of weakly interacting particles like neutrinos or axions, which is the dominant cooling process of the compact stars [40,41]. Since the emission rate depends on the couplings of those particles, it is important to understand how the interaction and the coupling of neutrinos or axions change in dense quark matter.

Neutrinos interact with quarks by the exchange of neutral currents, which is described, at low energy, as four-Fermi interaction,

$$\mathcal{L}_{\nu q} = \frac{G_F}{\sqrt{2}} \bar{\Psi}_L \gamma^\mu \Psi_L \bar{\nu}_L \gamma_\mu \nu_L, \quad (7.1)$$

where  $G_F = 1.166 \times 10^{-5} \text{ GeV}^{-2}$  is the Fermi constant. Again, by decomposing the quark fields as in Eq. (2.5) and integrating out  $\psi_-$  modes, the four-Fermi interaction becomes

$$\mathcal{L}_{\nu q} = \frac{G_F}{\sqrt{2}} \sum_{\vec{v}_F} \psi_{+L}^\dagger(\vec{v}_F, x) \psi_{+L}(\vec{v}_F, x) \bar{\nu}_L(x) \not{V} \nu_L(x) + \dots, \quad (7.2)$$

where the ellipsis denotes the higher order terms in the power expansion of  $1/\mu$ . Since the four-fermion interaction of quarks with opposite momenta gets enhanced a lot at low energy, as we have seen in the previous section (Sec. V), it may have significant corrections to the couplings of those weakly interacting particles to quarks. We first calculate the one-loop correction to the neutrino-quark four-Fermi coupling by the marginal four-quark interaction:

$$\begin{aligned} \delta \mathcal{L}_{\nu q} &= \frac{G_F}{\sqrt{2}} \psi_{+L}^\dagger(\vec{v}_F, x) \psi_{+L}(\vec{v}_F, x) \bar{\nu}_L(x) \not{V} \nu_L(x) \\ &\quad \times \frac{ig_3}{2M^2} \delta_{tv;us}^A \int_y \left[ \bar{\psi}_t(\vec{v}_F', y) \gamma^0 \psi_s(\vec{v}_F', y) \bar{\psi}_v(-\vec{v}_F', y) \gamma^0 \psi_u(-\vec{v}_F', y) \right] \\ &= \frac{4}{3} \frac{g_3}{2\pi} \frac{G_F}{\sqrt{2}} \psi_{+L}^\dagger(\vec{v}_F, x) \psi_{+L}(\vec{v}_F, x) \bar{\nu}_L(x) \bar{V} \cdot \gamma \nu_L(x), \end{aligned} \quad (7.3)$$

where  $\vec{v}_F$  and  $\vec{v}_F'$  are summed over and  $g_3$  is the value of the marginal four-quark coupling at the screening mass scale  $M$ . Now, in fact, because of the kinematical constraint due to the presence of the Fermi surface, only the cactus diagrams, shown in Fig. 5, contribute to the coupling corrections, which can be summed up as

$$\begin{aligned} \delta \mathcal{L}_{\nu q} &= \frac{4G_F}{3\sqrt{2}} \sum_{\vec{v}_F} \left[ \left( \frac{g_3}{2\pi - g_3} \right) \psi_{+L}^\dagger(\vec{v}_F, x) \psi_{+L}(\vec{v}_F, x) \nu_L^\dagger(x) \nu_L(x) \right. \\ &\quad \left. - \left( \frac{g_3}{2\pi + g_3} \right) \psi_{+L}^\dagger(\vec{v}_F, x) \psi_{+L}(\vec{v}_F, x) \bar{\nu}_L(x) \vec{v}_F \cdot \vec{\gamma} \nu_L(x) \right]. \end{aligned} \quad (7.4)$$

Similarly, for axions which couple to quarks as

$$\mathcal{L}_{aq} = \frac{1}{2f_{PQ}} \partial_\mu a \bar{\Psi} \gamma^\mu \gamma_5 \Psi, \quad (7.5)$$

where  $a$  is the axion field and  $f_{PQ}$  is the axion decay constant, we find the correction to the axion-quark coupling to be

$$\delta\mathcal{L}_{aq} = \frac{2}{3f_{PQ}} \sum_{\vec{v}_F} \left[ \left( \frac{g_3 \partial_0 a}{2\pi - g_3} \right) \psi_+^\dagger(\vec{v}_F, x) \gamma_5 \psi_+(\vec{v}_F, x) - \left( \frac{g_3 \vec{v}_F \cdot \vec{\nabla} a}{2\pi + g_3} \right) \psi_+^\dagger(\vec{v}_F, x) \gamma_5 \psi_+(\vec{v}_F, x) \right].$$

Therefore, as the marginal four-quark coupling approaches  $2\pi$ , the quark-neutrino and quark-axion couplings become divergent. Since at low energy  $g_3$  is quite large in dense quark matter, we argue that quark matter will produce neutrinos or axions copiously if the density of quark matter is high enough ( $\mu \gg \Lambda_{\text{QCD}}$ ) such that the marginal four-quark interaction gets enhanced sufficiently.

## VIII. CONCLUSION

We have studied in detail an effective field theory of QCD at high density, constructed by integrating out the anti-quarks to describe the low energy dynamics of dense quark matter. In the effective theory, the dynamics of quarks is effectively (1+1)-dimensional; the energy of quarks does not depend on the perpendicular momentum to the Fermi velocity, which just serves to label the degeneracy. At energy lower than the screening mass of electric gluons, the relevant interactions are four-Fermi interactions with opposite incoming momenta and the coupling with magnetic gluons.

Because of the dimensional reduction at high density, both four-Fermi interaction and magnetic gluon exchange interaction lead to Cooper-pair condensate of quarks in color anti-triplet channel for arbitrarily weak coupling. In the ladder approximation, the gap formed by the magnetic gluon exchange interaction is much bigger than the one by four-Fermi interaction. The color-flavor locking condensate is found to be energetically more preferred for high density quark matter with three light flavors, because the color-flavor locking gap is bigger than the spin one gap, if we include the four-Fermi interactions. Further, including the  $1/\mu$  corrections and the temperature effects, the critical density and the critical temperature for color superconducting phase are calculated by solving the gap equation in the HDL approximation.

Finally, we have calculated the corrections to the quark-neutrino four-Fermi interaction and to the quark-axion coupling in dense quark matter due to the marginal four-quark interaction. And we found the correction is quite significant and thus dense quark matter will copiously produce neutrinos and axions.

## ACKNOWLEDGMENTS

The author is grateful to Sekhar Chivukula, Andy Cohen, Raman Sundrum for useful comments, and wishes to thank Roman Jackiw, Steve Hsu, Cristina Manuel, Mannque Rho, Dirk Rischke, Tom Schäfer, and Andrei Smilga for enlightening discussions. The author wishes to acknowledge the financial support of the Korea Research Foundation made in the program year of 1998 (1998-15-D00022). This work was also supported by Pusan National University Research grant, 1998.

## REFERENCES

- [1] S. Weinberg, *Physica* **A96** (1979) 327; J. Gasser and H. Leutwyler, *Ann. Phys. (N.Y.)* **158** (1984) 142.
- [2] For pedagogical lectures on effective field theories, see D. B. Kaplan, “Effective Field Theories”, *Lectures given at the Seventh Summer School in Nuclear Physics*, Institute for Nuclear Theory, June 19-30, 1995, [nucl-th/9506035](#); A. Manohar, *Effective Field Theories*, *Lectures presented at the Lake Louise Winter Institute*, February, 1995, [hep-ph/9508245](#).
- [3] J. Polchinski, in *Proceedings of the 1992 TASI*, eds. J. Harvey and J. Polchinski (World Scientific, Singapore 1993).
- [4] D. Bailin and A. Love, *Phys. Rep.* **107**, 325 (1984).
- [5] R. Rapp, T. Schafer, E.V. Shuryak, and M. Velkovsky, *Phys. Rev. Lett.* **81**, 53 (1998).
- [6] M. Alford, K. Rajagopal, and F. Wilczek, *Phys. Lett. B* **422**, 247 (1998), [hep-ph/9711395](#); J. Berges and K. Rajagopal, *Nucl. Phys. B* **537** (1999) 443, [hep-ph/9804403](#).
- [7] M. Alford, K. Rajagopal, and F. Wilczek, *Nucl. Phys. B* **538** (1999) 443, [hep-ph/98044233](#).
- [8] N. Evans, S. Hsu, and M. Schwetz, *Nucl. Phys. B* **551** (1999) 275, [hep-ph/9808444](#); *Phys. Lett. B* **449** (1999) 281, [hep-ph/9810514](#).
- [9] K. Langfeld and M. Rho, *Quark condensation, Induced symmetry breaking and Color superconductivity at high density*, [hep-ph/9811227](#).
- [10] D.T. Son, *Phys. Rev. D* **59** (1999) 094019, [hep-ph/9812287](#).
- [11] D. K. Hong, *Phys. Lett. B* **473** (2000) 118, [hep-ph/9812510](#).
- [12] D.K. Hong, V.A. Miransky, I.A. Shovkovy, and L.C.R. Wijewardhana, *Phys. Rev. D* **61** (2000) 056001, [hep-ph/9906478](#).
- [13] T. Schäfer and F. Wilczek, *Phys. Rev. D* **60** (1999) 114033, [hep-ph/9906512](#).
- [14] R. Pisarski and D. Rischke, [nucl-th/990741](#); [nucl-th/9910056](#).
- [15] R. Pisarski and D. Rischke, *Phys. Rev. Lett.* **83** (1999) 37, [nucl-th/9811104](#).
- [16] J. Kapusta and T. Toimela, *Phys. Rev. D* **37** (1988) 3731; *ibid.* **39** (1989) 3197.
- [17] C. Manuel, *Phys. Rev. D* **53** (1996) 5866.
- [18] G. Alexanian and V.P. Nair, *Phys. Lett. B* **390** (1997) 370.
- [19] T. Schäfer and F. Wilczek, *Phys. Rev. Lett.* **82** (1999) 3956, [hep-ph/9811473](#); *Phys. Rev. D* **60** (1999) 074014, [hep-ph/9903503](#).
- [20] M. Luke and A. V. Manohar, *Phys. Lett. B* **286** (1992) 348.
- [21] D. K. Hong, *Phys. Rev. D* **57**, 3759 (1998), [hep-ph/9707432](#).
- [22] D. K. Hong, Y. Kim, and S. J. Sin, *Phys. Rev. D* **54** (1996) 7879.
- [23] V. Gusynin, D. K. Hong, and I. Shovkovy, *Phys. Rev. D* **57** (1998) 5230.
- [24] J. Kapusta, *Field Theory at Finite Temperature and Density* (Cambridge University Press, Cambridge, England, 1989) and references therein.
- [25] R. Pisarski, *Phys. Rev. Lett.* **63** (1989) 1129; E. Braaten and R. Pisarski, *Nucl. Phys. B* **337** (1990) 569; J. Frenkel and J. C. Taylor, *Nucl. Phys. B* **334** (1990) 119.
- [26] M. Alford, J. Berges, and K. Rajagopal, *Nucl. Phys. B* **558** (1999) 219, [hep-ph/9903502](#); R. D. Pisarski and D. H. Rischke, [nucl-th/9907094](#).
- [27] J.M. Cornwall, R. Jackiw, and E. Tomboulis, *Phys. Rev.* **2428** (1974); K. S. Tam, *Phys. Lett. B* **152** (1985) 238; See also M. Peskin, *Recent Developments in Quantum Field*

- Theory and Statistical Mechanics*, (Les House Summer School on Theoretical Physics, Aug. 1982).
- [28] N. Evans, J. Hormuzdiar, S.D.H. Hsu, and M. Schwetz, [hep-ph/9910313](#).
  - [29] S. D. H. Hsu, and M. Schwetz, [hep-ph/9908310](#).
  - [30] V.P. Gusynin, V.A. Miransky, and I.A. Shovkovy, Phys. Rev. D **52** (1995) 4747; Nucl. Phys. B **462** (1995) 249; C. N. Leung, Y. J. Ng, and A. W. Ackley, Phys. Rev. D **54** (1996) 4181; D.-S. Lee, C.N. Leung, and Y.J. Ng, Phys. Rev. D **55** (1997) 6504.
  - [31] V.P. Gusynin, V.A. Miransky, and I.A. Shovkovy, Phys. Rev. Lett. **83** (1999) 1291, [hep-ph/9811079](#).
  - [32] J. Berges and K. Rajagopal, Nucl. Phys. B **538** (1999) 215, [hep-ph 9804233](#); M.A. Halasz, *et. al* Phys. Rev. D **58** (1998) 096007, [hep-ph 9804290](#).
  - [33] M. Stephanov, K. Rajagopal, and E. Shuryak, Phys. Rev. Lett. **81** (1998) 4816, [hep-ph/9806219](#).
  - [34] J. Ellis, *et. al* Phys. Rev. D **54** (1996) 6986; M. Schmelling, *International Conference on High-Energy Physics* (ICHEP 96), Warsaw, Poland, 25-31 (Jul 1996) [hep-ex/9701002](#).
  - [35] R. Rapp, *et. al* [hep-ph/9904353](#) .
  - [36] T. Matsubara, Prog. Theor. Phys. **14**, 351 (1955); A.A. Abrikosov, L.P. Gorkov, and I.E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Pergamon Press, Oxford 1965).
  - [37] V.P. Gusynin and I.A. Shovkovy, Phys. Rev. D **56** (1997) 5251.
  - [38] W. E. Brown, J. T. Liu, and H.-c. Ren, [hep-ph/9912409](#).
  - [39] D. K. Hong, Talk given at TMU-Yale Symposium on *Dynamics of Gauge Fields*, December 13-15, 1999, Tokyo.
  - [40] For a review, see M. S. Turner, Phys. Rep. **197** (1990) 67; G. Raffelt, *ibid.* **198** (1990) 1; *Stars as Laboratories for Fundamental Physics* (University of Chicago Press: Chicago, 1996).
  - [41] D. K. Hong, Phys. Lett. B **445** (1998) 36.
  - [42] See for instance J. R. Schrieffer, *Theory of Superconductivity* (New York, W.A. Benjamin, 1964).

# FIGURES

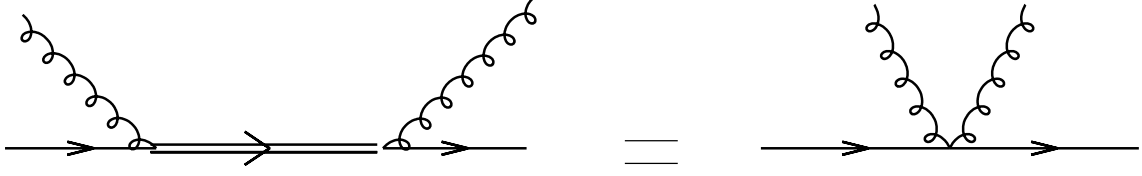


FIG. 1. The tree-level matching condition. Wiggly lines denote gluons; solid lines, states near the Fermi surface; and double solid line, states in the Dirac sea.

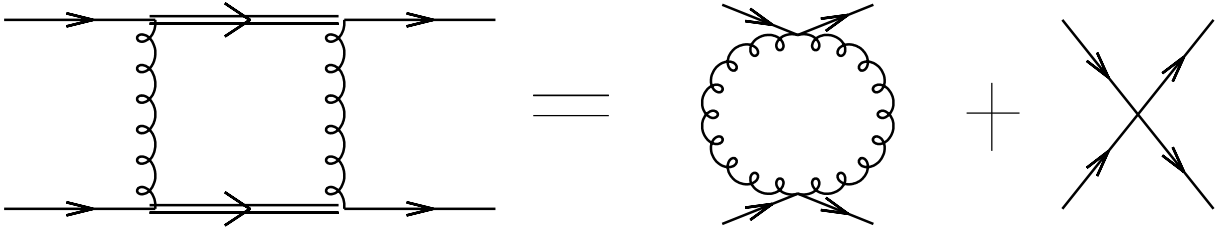


FIG. 2. The one-loop matching condition for a four-quark amplitude.

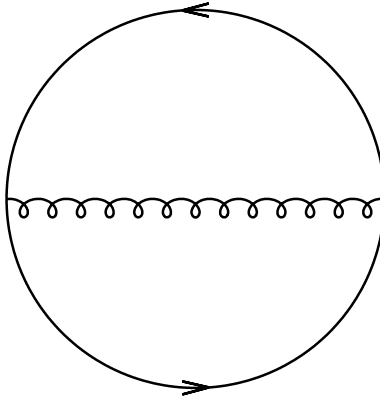


FIG. 3. The 2PI diagram in the leading order HDL approximation.

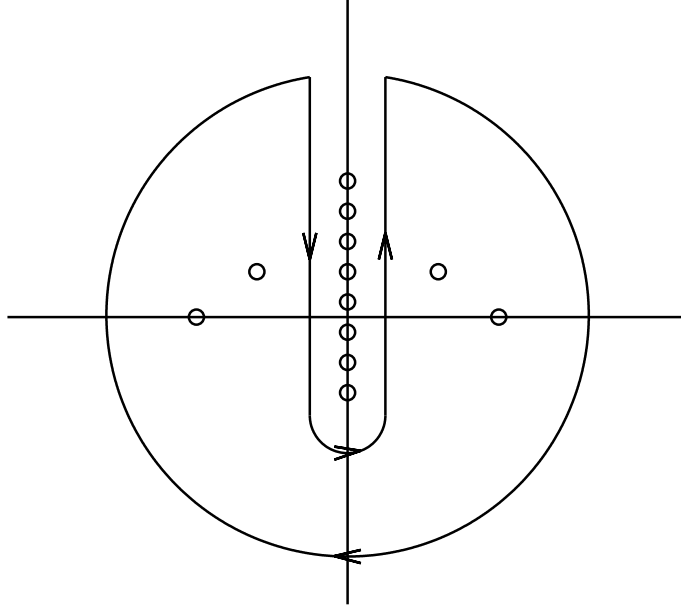


FIG. 4. The contour for integration over the Matsubara frequency. The circles are poles.

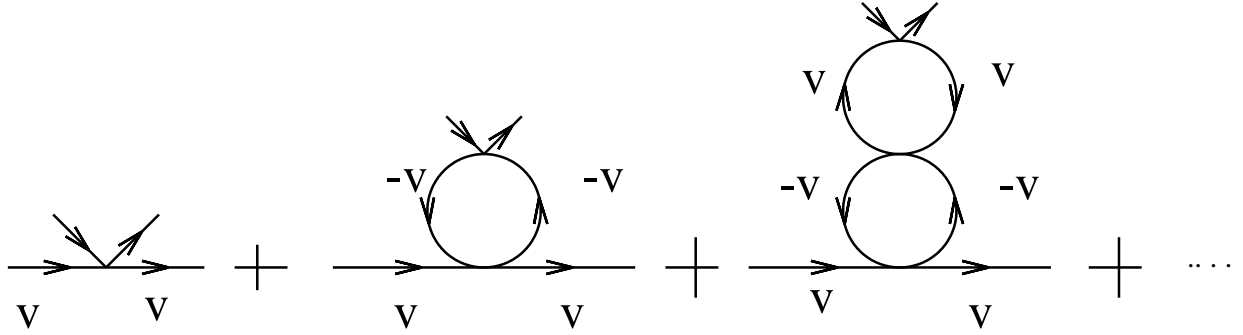


FIG. 5. The corrections to the quark-neutrino four-Fermi coupling.